Three-loop QCD corrections and b-quark decays

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We present three-loop next-to-next-to-next-to-leading order corrections to the heavy-to-heavy quark transitions in the limit of equal initial and final quark masses. In analogy with the previously found next-to-next-to-leading order corrections, the bulk of the result is due to the $\beta_0^2 \alpha_s^3$ corrections. The remaining genuine three-loop effects for the semileptonic $b \to c$ decays are estimated to increase the decay amplitude by 0.2(2)%. The perturbative series for the heavy-heavy axial current converges very well.

I. INTRODUCTION

The b quark decays most frequently into another relatively heavy quark, the c. These decays, especially when semileptonic, provide insight into the properties of heavy hadrons and an opportunity to determine standard model parameters such as the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{cb} [1].

In order to achieve good accuracy in those studies, one needs to know to what extent gluon emissions modify the weak transition amplitude. The evaluation of such effects is complicated by the presence of three energy scales: the masses of the radiating b and c quarks, and the momentum transfer to the leptons emitted in the decay process. The effects of a single gluon [next-to-leading order (NLO)] can be determined as an analytical function of all three scales (see, e.g., [2–4] and the references cited therein), but in the next-to-next-to-leading order (NNLO), an analytic expression is known only in the limiting case in which the c quark remains at rest with respect to the b quark [5], known as the zero-recoil configuration. It corresponds to the emission of the charged lepton and the neutrino in opposite directions.

Beyond the zero-recoil limit, numerous expansions have been employed to study semileptonic b decays with NNLO accuracy [6–8]. Those results improved our knowledge of the $b \to c$ decay rate, the CKM parameter V_{cb}, and in the future may also contribute to the determination of V_{ub}.

In the present paper we study effects of the next-to-next-to-next-to-leading order (NNNLO) to the transition of one heavy quark into another. Effects in this order, $\mathcal{O}(\alpha_s^3)$, have never before been calculated for any charged particle decay. The main reason is that the NNNLO calculations, especially with massive particles in intermediate states, are very difficult.

Here, we consider a particular kinematical limit in which the masses of the initial and final-state quarks are equal. This is the extreme zero-recoil limit: there is

no momentum transfer into leptons and no real gluon radiation. The zero-recoil configuration can be realized also with unequal quark masses, when the leptons, emitted antiparallel to each other, carry away all released energy. Radiative corrections in such configuration also consist of virtual gluon exchanges only. If the mass difference between the quarks is not very large, loop diagrams can be evaluated by expanding in this mass difference, around the equal mass ("extreme zero-recoil") limit considered in this paper.

Taking equal quark masses greatly simplifies the NNNLO calculation because we deal with two-point Green functions, similar to a quark propagator, with zero-momentum insertions (see Fig. 1). In fact, the technically related problems of the three-loop anomalous magnetic moment, Dirac form-factor, and heavy quark renormalization constants have already been solved [9–11].

For the present project, we modified a program originally written for the latter two studies [12]. Thus, with relatively little effort, we can obtain the first NNNLO correction to a heavy-to-heavy quark transition and confirm the expectations of its moderate size. It should be mentioned that despite the special kinematic limit in which we work, our result may help estimate NNNLO effects in a variety of other configurations because most of the kinematic effects reside in easily determined phase space effects, while the matrix elements depend on kine-
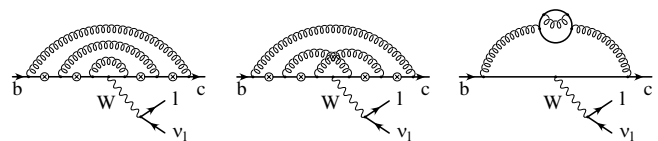

FIG. 1. Examples of three-loop corrections to the semileptonic $b \to c$ decay. The W can also be emitted from the quark line in the points denoted by crossed circles. This figure shows only examples of Abelian corrections, similar to photonic vertex corrections. The complete list of such diagrams is given in [22]. Non-Abelian diagrams are shown in subsequent figures.

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matics only relatively weakly. (There are exceptions to this picture; for example, along the zero-recoil line, the matrix elements are affected by a mass-singularity effect, manifesting itself in logarithms of the initial- and final-state quark mass ratio [13,14]. However, the physical ratio of b and c quark masses is not very large and our result obtained in the equal mass limit is expected to hold with reasonable accuracy [15].)

II. CORRECTIONS TO THE HEAVY-TO-HEAVY TRANSITION $b \to c l \nu$

The effect of QCD corrections on the $W \bar{c} b$ vertex is parametrized using two functions,

$$\gamma_\mu(1 - \gamma_5) \to \gamma_\mu[\eta_V(q^2) - \eta_A(q^2)\gamma_5]. \quad (1)$$

In the extreme zero-recoil limit we are considering, the momentum transfer into the W vanishes. In this case, the vector part of the interaction does not receive any QCD corrections,

$$\eta_V(q^2 = 0) = 1. \quad (2)$$

This is a useful check: the vector coupling part of the sum of all our three-loop diagrams, after the quark wave function renormalization, must vanish.

The axial part of the interaction does receive finite corrections even in the $q^2 = 0$ limit. They are expressed as a power series in α_s,

$$\eta_A(q^2 = 0) \equiv \eta_A$$
$$= 1 + \frac{\alpha_s}{\pi}C_F\eta_A^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 C_F\eta_A^{(2)}$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^3 C_F\eta_A^{(3)} + \mathcal{O}(\alpha_s^4). \quad (3)$$

Throughout this paper, we define α_s in the $\overline{\text{MS}}$ scheme at the scale of the quark mass, and use the pole definition of the quark mass.

The first two corrections to η_A have been known for a long time [14,16,17],

$$\eta_A^{(1)} = -\frac{1}{2},$$

$$\eta_A^{(2)} = C_F\left(-\frac{373}{144} + \frac{\pi^2}{6}\right) + (C_A - 2C_F)\left(-\frac{143}{144} - \frac{\pi^2}{12}\right.$$
$$+ \frac{\pi^2 \ln 2}{6} - \frac{1}{4}\zeta_3\right) + N_H T_R\left(\frac{115}{36} - \frac{1}{3}\pi^2\right) \quad (4)$$
$$+ \frac{7}{36}N_L T_R.$$

The SU(3) factors are $C_F = 4/3$, $C_A = 3$, and $T_R = 1/2$, and $N_{H,L}$ denotes the numbers of heavy and light quark flavors. For $b \to c$ decays, $N_H = 2$ and $N_L = 3$. The two- and three-loop corrections are expressed in terms of the Riemann zeta function and a tetra-logarithm,

$$\zeta_3 \simeq 1.202056903, \qquad \zeta_5 \simeq 1.036927755,$$
$$a_4 \equiv \text{Li}_4\left(\frac{1}{2}\right) \simeq 0.5174790617. \quad (5)$$

The result of the study reported here is the third correction, $\eta_A^{(3)}$. We find

$$\eta_A^{(3)} = N_H N_L T_R^2\left(-\frac{425}{162} + \frac{7}{27}\pi^2\right) + N_H T_R C_A\left(\frac{1339}{72} + \frac{7}{36}\zeta_3\pi^2 - \frac{121}{27}\zeta_3 - \frac{5}{4}\zeta_5 + \frac{136}{27}\pi^2\ln 2 + \frac{1}{3}\pi^2\ln^2 2 - \frac{1019}{243}\pi^2\right.$$
$$- \frac{53}{1080}\pi^4 - \frac{1}{3}\ln^4 2 - 8a_4\right) + N_H T_R C_F\left(\frac{7679}{2592} - \frac{355}{108}\zeta_3 + \frac{16}{27}\pi^2\ln 2 - \frac{4}{9}\pi^2\ln^2 2 - \frac{433}{486}\pi^2 + \frac{1}{30}\pi^4 + \frac{4}{9}\ln^4 2\right.$$
$$+ \frac{32}{3}a_4\right) + N_H^2 T_R^2\left(-\frac{1055}{324} + \frac{8}{3}\zeta_3\right) + N_L T_R C_A\left(\frac{469}{648} + \frac{19}{18}\zeta_3 - \frac{5}{27}\pi^2\ln 2 + \frac{2}{27}\pi^2\ln^2 2 + \frac{49}{216}\pi^2 - \frac{11}{648}\pi^4\right.$$
$$+ \frac{1}{27}\ln^4 2 + \frac{8}{9}a_4\right) + N_L T_R C_F\left(\frac{2293}{864} - \frac{23}{18}\zeta_3 + \frac{10}{27}\pi^2\ln 2 - \frac{4}{27}\pi^2\ln^2 2 - \frac{14}{27}\pi^2 + \frac{11}{324}\pi^4 - \frac{2}{27}\ln^4 2 - \frac{16}{9}a_4\right)$$
$$+ N_L^2 T_R^2\left(\frac{25}{324} - \frac{1}{27}\pi^2\right) + C_A^2\left(-\frac{16241}{5184} + \frac{11}{144}\zeta_3\pi^2 + \frac{215}{144}\zeta_3 - \frac{5}{3}\zeta_5 + \frac{139}{108}\pi^2\ln 2 - \frac{25}{108}\pi^2\ln^2 2 - \frac{1423}{1728}\pi^2\right.$$
$$+ \frac{97}{6480}\pi^4 - \frac{2}{27}\ln^4 2 - \frac{16}{9}a_4\right) + C_F C_A\left(-\frac{2723}{864} + \frac{1}{18}\zeta_3\pi^2 - \frac{79}{24}\zeta_3 + \frac{65}{12}\zeta_5 - \frac{215}{108}\pi^2\ln 2 + \frac{10}{27}\pi^2\ln^2 2 + \frac{467}{288}\pi^2\right.$$
$$- \frac{359}{6480}\pi^4 + \frac{13}{54}\ln^4 2 + \frac{52}{9}a_4\right) + C_F^2\left(-\frac{1141}{576} + \frac{7}{12}\zeta_3\pi^2 + \frac{40}{9}\zeta_3 - \frac{20}{3}\zeta_5 - \frac{7}{6}\pi^2\ln 2 + \frac{5}{27}\pi^2\ln^2 2 + \frac{155}{216}\pi^2\right.$$
$$- \frac{7}{216}\pi^4 - \frac{5}{27}\ln^4 2 - \frac{40}{9}a_4\right). \quad (6)$$
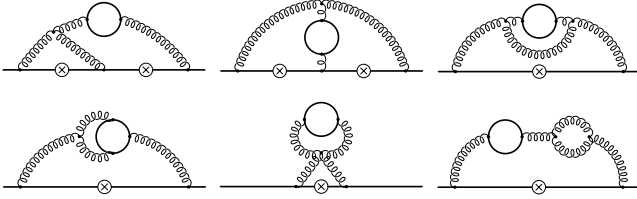
FIG. 2. Three-loop non-Abelian diagrams with fermion loops. Solid line closed loops denote light or heavy quarks. As in the previous figure, the *W* boson can be emitted at any vertex denoted by a crossed circle.

Among these ten contributions, differing by SU(3) factors, the first seven come from diagrams containing one or two fermion loops, with light and/or heavy quarks. Among them, the second and the fifth arise from non-Abelian diagrams with three- or four-gluon vertices, shown in Fig. 2. The remaining five contributions containing fermion loops are analogous to those in QED, with one example shown in Fig. 1.

The last three contributions in Eq. (6) correspond to diagrams without closed fermion loops. The first and the second of them are non-Abelian, and receive contributions from the diagrams shown in Fig. 3, as well as from the nonplanar diagrams without multigluon vertices. The latter, QED-like diagrams, also give the last contribution in Eq. (6).

In addition to the vertex correction, the wave-function renormalization also contributes to $\eta_A^{(3)}$. The $\mathcal{O}(\alpha_s^3)$ part of the on-shell quark wave-function renormalization constant Z_2 was found in [11]. To our knowledge, the present study is the first application of that result.

One novel feature of the on-shell constant Z_2 at the NNNLO is that some of the non-Abelian contributions are gauge-dependent in dimensional regularization [11]. In contrast, dimensionally regularized on-shell Z_2 in QED is gauge-independent to all orders, and before the explicit calculation in [11], it had been conjectured that the QCD result might also be gauge-independent [18]. The calculation presented here shows that the gauge dependence of Z_2 found in [11] is needed to make the full vertex correction $\eta_{A,V}^{(3)}$ gauge invariant. The cancellation of the gauge parameter dependence is a useful check, in addition to the vanishing of $\eta_V^{(3)}$.

III. NUMERICAL RESULTS AND SUMMARY

Substituting the numerical values of the coefficients of the SU(3) factors, we find for the first three corrections to η_A

$$\eta_A^{(1)} = -0.5, \qquad \eta_A^{(2)} \simeq -0.976C_A + 1.01C_F - 0.0954N_H T_R + 0.194N_L T_R,$$

$$\eta_A^{(3)} \simeq -2.06C_A^2 + 0.958C_A C_F - 2.15C_F^2 - 0.124C_A N_H T_R + 1.03C_F N_H T_R + 2.13C_A N_L T_R + 0.202C_F N_L T_R$$
$$- 0.0507N_H^2 T_R^2 - 0.0647N_H N_L T_R^2 - 0.288N_L^2 T_R^2. \tag{7}$$

For the case of the *b*-quark decay into a *c* quark ($N_L = 3, N_H = 2$), we get in the extreme zero-recoil limit,

$$\eta_A \simeq 1 - 0.667\frac{\alpha_s}{\pi} - 1.85\left(\frac{\alpha_s}{\pi}\right)^2 - 11.1\left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4) \simeq 1 - 0.0510 - 0.0108 - 0.00495 \simeq 0.933. \tag{8}$$

In the last two lines we substituted $\alpha_s = 0.24$, roughly (to within 20%) corresponding to the value appropriate for the scale $\sqrt{m_b m_c}$. We see that the three-loop calculation presented here contributes about 7% of the deviation of η_A from unity, and decreases η_A by about half of a percent. A large part of this NNNLO correction is due to the known effects that can be absorbed in the running of the coupling constant, that is contributions of the type $\alpha_s^3 \beta_0^2$, with $\beta_0 = 11 - \frac{2}{3}N_F$, and N_F denoting the number of fermions contributing to the running. This is often referred to as the Brodsky-Lepage-Mackenzie (BLM) correction [19]. We can find it using the coefficient

of N_L^2 (diagrams containing two light fermion loops) from Eq. (6). We obtain, taking $N_F = 4$ quarks in the running,

$$C_F \eta_A^{(3)}(\text{BLM}) = \left(4 - \frac{33}{2}\right)^2 C_F T_R^2\left(\frac{25}{324} - \frac{\pi^2}{27}\right) = -15.0. \tag{9}$$

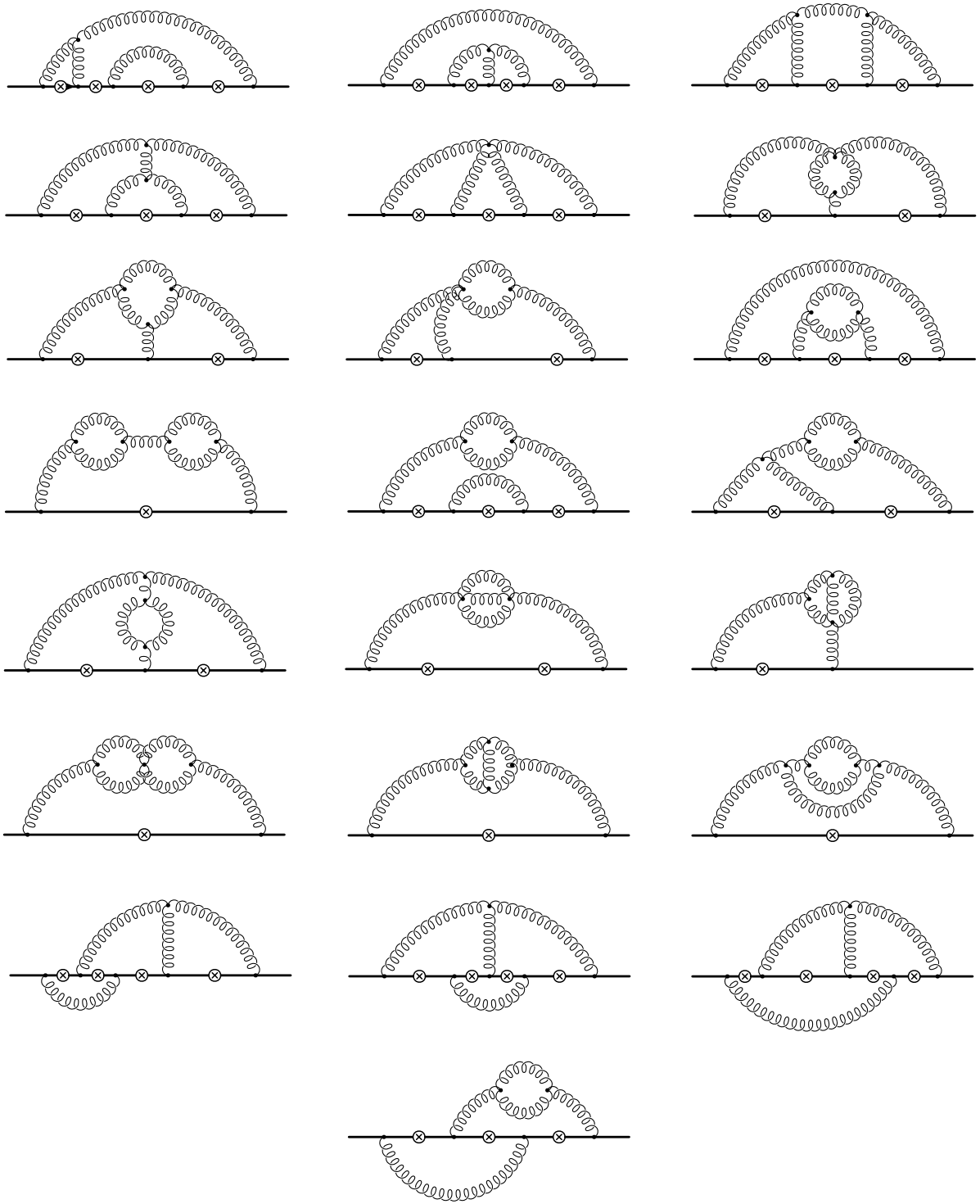We see that the remaining "genuine" three-loop correc-

FIG. 3. Three-loop non-Abelian gluonic diagrams.

tion is small,

$$C_F \eta_A^{(3)}(\text{non-BLM}) = -11.1 + 15.0 = 3.9. \qquad (10)$$

It is possible to extend this result beyond the limit of zero-momentum transfer to the leptons by expand-

ing in the difference of the heavy quark masses. However, we believe that our result estimates the true value of the correction at physical values of m_b and m_c to within at most a factor of 2. The corrections due to unequal m_b and m_c are suppressed by two powers of $1 - m_c/m_b$ [15]. Given that the new non-BLM effect in

Eq. (10) increases η_A by about 0.2%, we believe that it is safe to estimate the corresponding effect for the real $b \rightarrow c$ decay at zero recoil as 0.2(2)%. Finally, we note that η_A determines corrections to the total semileptonic decay rate if the difference of the quark masses is relatively small [20] and can be used to estimate the $b \rightarrow c$-decay rate [21]. Such analysis will be presented elsewhere.

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